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Fuzzy linear programming for problems of water allocation under uncertainty

G. Tsakiris and M. Spiliotis
Lab. of Reclamation Works and Water Resources Management
School of Rural and Surveing Eng., National Technical University of Athens
9, Iroon Polytechniou 157 73 Athens – Greece.

Abstract: One of the basic problems of Water Resources Management is the allocation of water resources to various users in an optimal and equitable way respecting the constraints imposed by the environment. In the Systems Analysis formulations the problem of water allocation to various users has been treated as a simple linear programming problem with the objective of maximizing the total water productivity. In this type of formulation other social and environmental factors were introduced in the form of linear constraints. Unfortunately in the real world most of the parameters used are very uncertain. Therefore even if the linear programming simplification is accepted, neither the constraints, nor the expected revenues, can be characterized by certainty. In this work a fuzzy set representation of the unit revenue of each use together with a fuzzy representation of each set of constraints, are used to expand the capabilities of the linear programming formulation. Numerical examples are presented for illustrative purposes and useful conclusions are derived.

Key words: fuzzy numbers, linear programming, water allocation, water resources management.

1. INTRODUCTION

Although Integrated Water Resources Management (IWRM) is a continuous process, conventionally it is approached as a two – phase process, comprising the Strategic and the Operational management.

The objective of the first phase is the selection of measures and projects aiming at fulfilling the present and future water requirements while protecting the water resources and the environment thus securing conditions of sustainable development.

Strategic management is applied at a river basin or a number of adjacent basins incorporating all sites of water availability and water consumption together with all important elements of the environment.

The most serious difficulty the decision makers face at this strategic level is the uncertainty by which all the determinants are estimated for the future time horizons.

The paper presents a methodology for solving a category of problems known as “water allocation to various users’, using the theory of fuzzy numbers. Various cases are analyzed depending on the parameters which are considered as fuzzy numbers. In all cases the linear programming approach is used.

The background of the theory of fuzzy numbers can be found in a variety of books and papers (e.g. Klir and Yuan, 1995).

2. METHODOLOGY

2.1 Fuzzy linear programming (F.L.P)

A formulation of the F.L.P is the flexible approach in which the decision variables are crisp numbers while the right hand coefficients are characterized by uncertainties.

Another formulation of F.L.P is the Possibilistic Approach. In general, Possibilistic Programming deals with problems where the coefficients of decision variables are obtained as fuzzy numbers. However, this means that in an uncertainty environment a crisp decision is made to meet some decision criteria. Furthermore, it was proposed that the coefficient of decision variables should consist of crisp numbers, while the decision variables and the right hand coefficients should be fuzzy numbers.

2.2 Flexible Approach

The problem is formulated according to Zimmerman's formulation (1996) in which the objective function should be fuzzily greater than a number, while the constraints are fuzzy:

$$\begin{aligned}
&\sum_{i=1}^{n} c_i x_i \tilde{\geq} z \\
&\sum_{i=1}^{n} A_{ij} x_i \tilde{\leq} B_j \quad j = 1,\dots,m \\
&x_i \geq 0
\end{aligned} \tag{1}$$

where the symbols $\tilde{\leq}, \tilde{\geq}$ denote the relaxed fuzzy versions of the ordinary inequalities.

In this section the crisp numbers are selected as decision variables. In order to solve this problem the next steps are followed:

i. Estimation of the membership function for the objective function. A linear type membership function is assumed:

$$\mu_c\left(\sum_{i=1}^{n} c_i x_i\right) = \begin{cases} 0 & \text{if } \sum_{i=1}^{n} c_i x_i \leq z_{\underline{0}} \\ 1 & \text{if } \sum_{i=1}^{n} c_i x_i \geq z_0 \\ \dfrac{\sum_{i=1}^{n} c_i x_i - z_{\underline{0}}}{z_0 - z_{\underline{0}}} & \text{if } z_{\underline{0}} < \sum_{i=1}^{n} c_i x_i < z_0 \end{cases} \tag{2}$$

ii. Estimation of the membership functions for the constraints. Again linear type is assumed.

$$\mu_j\left(\sum_{i=1}^{n} A_{ji}x_i\right)=\begin{cases} 0 & \text{for } \sum_{i=1}^{n} A_{ji}x_i \geq B_j + t_j \\ 1 & \text{for } \sum_{i=1}^{n} A_{ji}x_i \leq B_j \\ 1-\dfrac{\sum_{i=1}^{n} A_{ji}x_i - B_j}{t_j} & \text{for } B_j < \sum_{i=1}^{n} A_{ji}x_i < B_j + t_j \end{cases} \tag{3}$$

where $t_j \geq 0$.

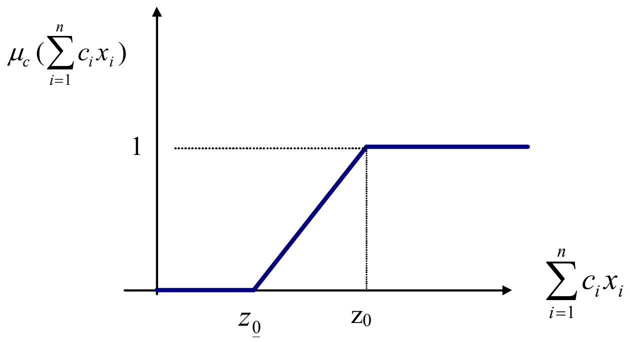

Figure 1. The linear membership function for the objective function.

iii. The min section is selected as the aggregation operator:

$$\max_{x\geq 0} \min[\mu_C(x), \mu_1(x), \ldots, \mu_j(x), \ldots, \mu_m(x)] \tag{4}$$

or equivalently by introducing the auxiliary variable α which indicates the common degree of satisfaction both of the fuzzy constraints and the objective function:

$$\begin{aligned} &\max \alpha \\ &\mu_c(x) \geq \alpha \\ &\mu_j(x) \geq \alpha, \quad j = 1,\ldots,m \\ &\alpha \in [0,1] \quad , \\ &x \geq 0 \end{aligned} \tag{5}$$

It is obvious that according to the above formulation the problem is symmetrical with respect to the fuzzy constraints and the objective function. Therefore this problem can be transformed into the following equivalent conventional linear programming problem:

$$\begin{aligned} &\max \alpha \\ &\sum_{i=1}^{n} c_i x_i \geq z_{\underline{0}} + \alpha(z_0 - z_{\underline{0}}) \\ &\sum_{i=1}^{n} A_{ji} x_i \leq B_j + (1-\alpha)\cdot t_j, \quad j = 1,\ldots,m \\ &\alpha \in [0,1] \\ &x_i \geq 0 \end{aligned} \tag{6}$$

and other crisp constraints.

2.3 Fuzzy Linear Programming with central triangular fuzzy variables

The second method of F.L.P. is due Tanaka et al. (2000). The method uses central triangular fuzzy numbers (c.t.f.n.) as decision variables (possibilistic programming). The problem formulation according to this methodology is as follows:

The objective function requires to be greater than a fuzzy number instead of being maximized:

$$\sum_{i=1}^{n} c_i \tilde{X}_i \geq \tilde{z} \tag{7}$$

under the constraints

$$\sum_{i=1}^{n} A_{ij} \tilde{X}_i \leq \tilde{B}_j, \; j = 1, \ldots, m \tag{8}$$

where $\tilde{z}$, $\tilde{B}_j$ are central triangular fuzzy numbers.

In this case the decision variables are c.t.f.n.:

$$\tilde{X}_i = (a_i, w_i)_T \text{, where } w_i \geq 0$$

where a_i and w_i are the centre and the spread of the c.t.f.n. $\tilde{X}_i$, respectively. Due to the fact that $\tilde{X}_i$ is a c.t.f.n. the following membership function is obtained:

$$\mu_{X_i}(x_i) = \begin{cases} 0 & \text{if } \; x_i \leq a_i - w_i, \quad x_i \geq a_i + w_i \\ 1 & \text{if } \; x_i = a_i \\ 1 - \dfrac{a_i - x_i}{w_i} & \text{if } \; a_i \geq x_i \geq a_i - w_i \\ 1 + \dfrac{a_i - x_i}{w_i} & \text{if } \; a_i \leq x_i \leq a_i + w_i \end{cases} \tag{9}$$

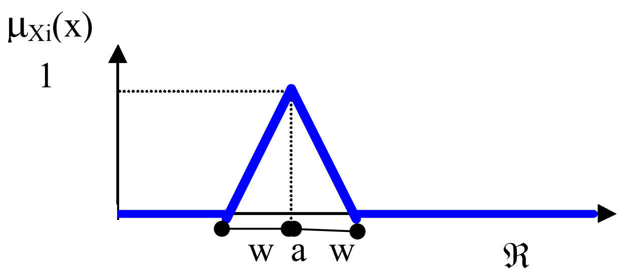

Figure 2. Central triangular fuzzy numbers.

Briefly the F.L.P. can be described as follows:

$$\begin{cases} \max f(a_i, w_i) \\ \sum_{i=1}^{n} c_i \tilde{X}_i \geq \tilde{z} \\ \sum_{i=1}^{n} A_{ij} \tilde{X}_i \leq \tilde{B}_j, \ j = 1,...,m \\ \tilde{X}_i \geq 0 \end{cases} \tag{10}$$

where $f(a_i, w_i)$ is a linear function of the centres and the spreads of c.t.f.n. and is called substitute objective function. This function enables us to determine the desired direction of the solution.

We observe that the model is fully symmetrical with respect to objective function and the constraints. This is even more obvious by substituting:

$$g_{ij} = \begin{pmatrix} -c_i \\ A_1 \\ \ldots \\ A_m \end{pmatrix}, \quad \tilde{D}_j = \begin{pmatrix} -\tilde{z} \\ \tilde{B}_1 \\ \ldots \\ \tilde{B}_m \end{pmatrix}, \qquad j = 1,..,m+1 \tag{11}$$

where $\tilde{D}_j$ is a c.t.f.n.:

$$\tilde{D}_j = \left(d_j, q_j\right)_T, \ q_j \geq 0 \tag{12}$$

The model becomes:

$$\begin{aligned} &\sum_{i=1}^{n} g_{ij} \tilde{X}_i \leq \tilde{D}_j \\ &\tilde{X}_i \geq 0 \end{aligned} \tag{13}$$

Given a fuzzy vector $x = \left(a, w\right)_T$ the fuzzy linear function $\sum_{i=1}^{n} g_{ij} \tilde{X}_i$ can be described as follows:

$$\left(\sum_{i=1}^{n} g_{ij} x_i \right)_T = \sum_{i=1}^{n} g_{ij} \tilde{X}_i = \left(\sum_{i=1}^{n} g_{ij} a_i \ , \ \sum_{i=1}^{n} \left|g_{ij}\right| w_i \right)_T \tag{14}$$

Definition: Let $A = (a, w_1)$ and $B = (b, w_2)$ c.t.f.n. A fuzzy inequality $A \leq B$ (according to Tanaka et al., 2000) is defined by the following two inequalities (Fig. 3):

$$\begin{aligned} &a + (1-h) \cdot w_1 \leq b + (1-h) w_2 \\ &a - (1-h) \cdot w_1 \leq b - (1-h) w_2 \end{aligned} \tag{15}$$

where $h \in \left[0,1\right]$ is a predetermined level.

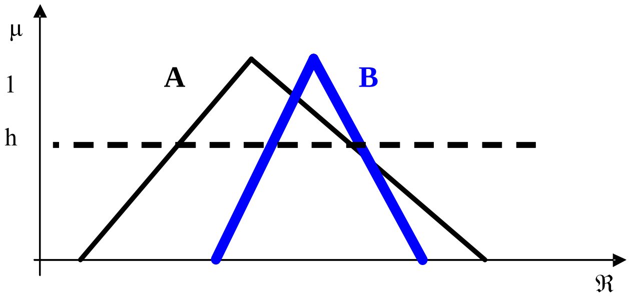

Figure 3. Explanation of inequality with the degree h.

According to the previous definition the j^{th} fuzzy constraint condition can be rewritten as follows:

$$(\sum_{i=1}^{n} g_{ij} x_i)_h \,\tilde{\leq}\, D_{j_h} \text{ , where } h \in [0,1] \tag{16}$$

or equivalently:

$$\begin{aligned} \sum_{i=1}^{n} g_{ij} a_i + (1-h) \cdot \sum_{i=1}^{n} |g_{ij}| w_i \leq d_j + (1-h) q_j \\ \sum_{i=1}^{n} g_{ij} a_i - (1-h) \cdot \sum_{i=1}^{n} |g_{ij}| w_i \leq d_j - (1-h) q_j \end{aligned} \tag{17}$$

where $j=1,\dots,m+1$ and $i=1,\dots,n$.

The linear function *f* is defined by the centres and the spreads of fuzzy decision variables:

$$\text{Max (or min) } k_1 \cdot \sum_{i=1}^{n} p_{ij} a_i + k_2 \cdot \sum_{i=1}^{n} |e_{ij}| w_i \text{ ,} \tag{18.a}$$

where k_1 and k_2 are the weights of the centres and the spreads of fuzzy decision variables. Moreover p_i and q_i must be greater than zero, p_i and q_i are the weights of the centre and the spread of the i^{th} fuzzy decision variable. The function f therefore, operates as utility function.

In case of water supply to different demands a complex utility function f may be established based on the authority of each decision maker as follows:

$$f = \sum_{i=1}^{I} K_i f_i(a, W) \tag{18.b}$$

where:

K_i: the assigned weight to agency i

I: the number of water demands

$f_i(a, W)$: the individual functions of each agency i indicates the objective function of agency i in imposing allocated water demand.

In the case that h=0, the problem is equivalent to the interval linear programming problem as described by Tanaka et al. (2000).

3. NUMERICAL EXAMPLES

3.1 The crisp problem of water allocation

A simple water system is presented in Figure 4. It is comprised of 6 simple surface water resources nodes NO_1, NO_2, N_1, N_2, N_3, N_4. The consumption centres A_1 and A_2 are receiving water from the nodes N_1 and N_2, respectively. It is required to calculate the quantity that is transferred from N_1 to the centre 1 and the quantity from N_2 to the centre 2 so that the total economic return is as high as possible and at the same time the minimum allowable quantities, for the downstream areas, are secured.

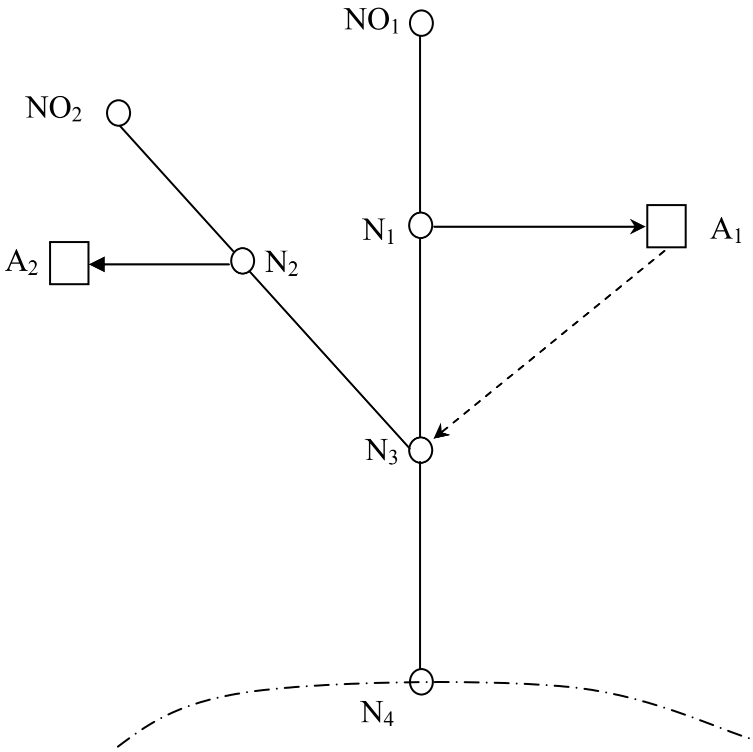

Figure 4. Schematic representation of the simple water system.

Data:

Volume of water available (during the period of analysis) in NO_1-N_1: 66 units of volume (UV)

Volume of water available (during the period of analysis) in NO_2-N_2: 59 UV

P_1, P_2 are the maximum water demands in the centre 1 and 2 respectively:

$P_1 = 49$ UV

$P_2 = 35$ UV

The percentage of the flow, which returns to N_3 from the centre 1, is estimated to be 30 % of the corresponding requirement of centre 1 after treatment.

Minimum allowable flows in the branches $N_3 - N_4$, $N_2 - N_3$ have been estimated 47 and 8 UV.

The revenue from each unit volume consumed in the centre 1 and centre 2 are 1 benefit units (BU) and 1.5 BU, respectively.

The formulation of the above simplified optimization problem is formulated as follows:

$$\begin{aligned}
&\max\ (x_1 + 1.5 \cdot x_2) \\
&x_1 \le 66 \\
&x_2 \le 59 \\
&x_1 \le 49 \\
&x_2 \le 35 \\
&125 - 0.7 \cdot x_1 - x_2 \ge 47 \\
&59 - x_2 \ge 8 \\
&x_1,\ x_2 \ge 0
\end{aligned} \tag{19}$$

Solving the L.P. problem, the allocation with the highest revenue will be as follows: $x_1 = 49\,\mathrm{UV}$, $x_2 = 35\mathrm{UV}$, achieving the maximum revenue, which is $z^* = 101.5\mathrm{BU}$.

3.2 The fuzzy problem of water allocation, in the case of crisp numbers as decision variables

Consider the same simplified water system as the one introduced before, but now assume that the numerical values of water maximum demands P_1, P_2 at the centres 1 and 2 and the minimum allowable volume requirements in the branches or the system can not be precisely defined. These parameters are only known as triangular fuzzy numbers:

$$\begin{aligned}
&\tilde{p}_1 = \langle 49, 52, 55 \rangle_T,\ \tilde{p}_2 = \langle 35, 37, 39 \rangle_T, \\
&Q\tilde{N}_1 = \langle 45, 47, 49 \rangle_T,\ Q\tilde{N}_2 = \langle 5, 8, 11 \rangle_T
\end{aligned} \tag{20}$$

Moreover it is assumed that the objective function is defined fuzzily with the form of a triangular fuzzy number:

$$\tilde{z} = \langle 75, 90, 105 \rangle_T \tag{21}$$

The quantities $125 - Q\tilde{N}_1$, $59 - Q\tilde{N}_2$ are also fuzzy numbers according to the extension principle:

$$\langle 125-49,\ 125-47, 125-45 \rangle_T,\ \langle 59-11, 59-8,\ 59-5 \rangle_T \tag{22}$$

Under this assumption the gray region of the constraint of the maximum water demand for the centre 1 is:

$$x_1 \in (49, 55) \tag{23}$$

For example the membership function of the constraint of the maximum water demand in the centre 1 becomes according to Eq.3 (Fig. 5):

$$\mu_{p1}\left(x_1\right)=\begin{cases}1 & \text{if } x_1 \le 49\\ 0 & \text{if } x_1 \ge 55\\ 1-\dfrac{x_1-49}{6} & \text{if } 49 \le x_1 \le 55\end{cases} \tag{24}$$

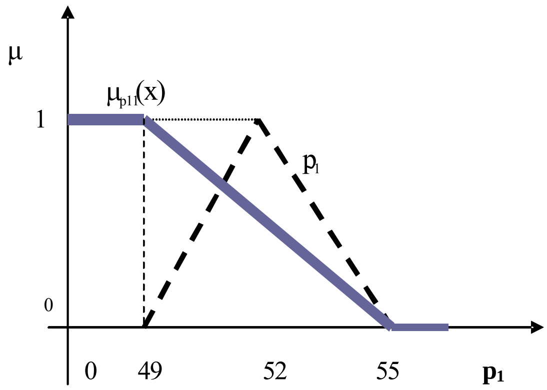

Figure 5. The fuzzy number $\tilde{p}_1$ and the membership function of the fuzzy constraint $\mu_{p1}(x_1)$ according to flexible approach.

The gray region of the constraint of the maximum water demand in the centre 2 is:

$$x_2 \in \left(35,39\right) \tag{25}$$

The gray region of the constraint of minimum allowable volume in the branches $N_3 - N_4$ and $N_2 - N_3$ are:

$$0.7x_1 + x_2 \in \left(125-49,125-45\right) \tag{26}$$

and

$$x_2 \in \left(59-11,\ 59-5\right)\text{, respectively} \tag{27}$$

The gray region of the objective function is:

$$1.5x_1 + x_2 \in \left(75,105\right) \tag{28}$$

According to Eq.6 the problem can be transformed into the following equivalent conventional L.P problem:

$$
\begin{aligned}
&\max \alpha \\
&(x_1 + 1.5 \cdot x_2) \geq 75 + 30 \cdot \alpha \\
&x_1 \leq 66 \text{ (crisp constraint)} \\
&x_2 \leq 59 \text{ (crisp constraint)} \\
&x_1 \leq 49 + (1-\alpha)6 \\
&x_2 \leq 35 + (1-\alpha)4 \\
&0.7 \cdot x_1 + x_2 \leq 76 + (1-\alpha)4 \\
&x_2 \leq 48 + (1-\alpha)6 \\
&x_1,\ x_2 \geq 0 \\
&\alpha \in [0,1]
\end{aligned}
\tag{29}
$$

The solution is:

$$
\begin{aligned}
&\alpha = 0.912 \\
&z^{**} = 102.5\,\text{BU} \\
&x_1 = 49.5\,\text{UV} \\
&x_2 = 35.34\,\text{UV}
\end{aligned}
\tag{30}
$$

3.3 The fuzzy problem of water allocation, in the case of fuzzy numbers as decision variables

In this section according to the section 2.3, the decision variables are selected to be c.t.f.n:

$$\tilde{X}_i = (a_i, w_i)_T,\ w_i \geq 0$$

Given the fuzzy triangular number $\tilde{X}_1, \tilde{X}_2$ the fuzzy linear function $\tilde{X}_1 + 1.5 \cdot \tilde{X}_2$ can be described as follows:

$$
\left\langle
\begin{matrix}
(a_1 + 1.5 \cdot a_2 - w_1 - 1.5 \cdot w_2),\ (a_1 + 1.5 \cdot a_2), \\
(a_1 + 1.5 \cdot a_2 + w_1 + 1.5 \cdot w_2)
\end{matrix}
\right\rangle_T
\tag{31}
$$

Given the fuzzy triangular number $\tilde{X}_1, \tilde{X}_2$ the fuzzy linear function $0.7 \cdot \tilde{X}_1 + \tilde{X}_2$ can be described as follows:

$$
\left\langle (0.7 \cdot a_1 + a_2 - 0.7 \cdot w_1 - w_2),\ (0.7 \cdot a_1 + a_2),\ (0.7 \cdot a_1 + a_2 + 0.7 \cdot w_1 + w_2) \right\rangle_T
\tag{32}
$$

At first we investigate the fuzzy linear problem with h=0 (interval linear programming). According to Eq.17 the above F.L.P can be written for h=0:

$$a_1 + 1.5 \cdot a_2 + w_1 + 1.5 \cdot w_2 \geq 105$$
$$a_1 + 1.5 \cdot a_2 - w_1 - 1.5 \cdot w_2 \geq 75 \tag{33}$$

$$a_1 + w_1 \leq 66$$
$$a_1 - w_1 \leq 66 \tag{34}$$

$$a_2 + w_2 \leq 59$$
$$a_2 - w_2 \leq 59 \tag{35}$$

$$a_1 + w_1 \leq 55$$
$$a_1 - w_1 \leq 49 \tag{36}$$

$$a_2 + w_2 \leq 39$$
$$a_2 - w_2 \leq 35 \tag{37}$$

$$(0.7 \cdot a_1 + a_2) + (0.7 \cdot w_1 + w_2) \leq 80$$
$$(0.7 \cdot a_1 + a_2) - (0.7 \cdot w_1 + w_2) \leq 76 \tag{38}$$

$$(a_2 + w_2) \leq 54$$
$$(a_2 - w_2) \leq 48 \tag{39}$$

The following cases for the substitute objective function can be selected:

- Case 1

The case of $k_2 = 0$ and $p_{ij} = c_i$ means that we wish to obtain the maximum centre of revenue:

$$\text{Max}\ (a_1 + 1.5a_2) \tag{40}$$

In addition the f operates as utility function, while the weights (Eq. 18b) was selected according to the revenue from each unit volume consumed and f_i was selected to be the central values of water consumed of each demand centre.

- Case 2

The case of $k_1 = 0$ and $e_{ij} = c_i$ means that we want to obtain the minimum spread of the profit:

$$\text{Min}(w_1 + 1.5w_2) \tag{41}$$

The results of the above cases are shown in Table 1.

Table 1. Solution for different cases of F.L.P. problem for h=0.

h=0	Max $(a_1+1.5a_2)$	Min $(w_1+1.5w_2)$
a_1	52.00	49.00
w_1	3.00	0
a_2	37.00	36.17
w_2	2.00	1.17
z^+	113.50	105
z^-	101.50	101.50
$a_1+1.5a_2$	107.50	103.25
$w_1+1.5w_2$	6.00	1.75

According to Eq.17 the constraints of the above F.L.P can be written for h = 0.5 which is a more crisp problem compared with h = 0:

$$\begin{aligned} & a_1 + 1.5 \cdot a_2 + 0.5\left(w_1 + 1.5 \cdot w_2\right) \geq 90 + 15 \cdot 0.5 \\ & a_1 + 1.5 \cdot a_2 - 0.5\left(w_1 + 1.5 \cdot w_2\right) \geq 90 - 15 \cdot 0.5 \end{aligned} \tag{42}$$

$$\begin{aligned} & a_1 + 0.5 w_1 \leq 66 \\ & a_1 - 0.5 w_1 \leq 66 \end{aligned} \tag{43}$$

$$\begin{aligned} & a_2 + w_2 \cdot 0.5 \leq 59 \\ & a_2 - w_2 \cdot 0.5 \leq 59 \end{aligned} \tag{44}$$

$$\begin{aligned} & a_1 + w_1 \cdot 0.5 \leq 52 + 3 \cdot 0.5 \\ & a_1 - w_1 \cdot 0.5 \leq 52 - 3 \cdot 0.5 \end{aligned} \tag{45}$$

$$\begin{aligned} & a_2 + w_2 \cdot 0.5 \leq 37 - 2 \cdot 0.5 \\ & a_2 - w_2 \cdot 0.5 \leq 37 - 2 \cdot 0.5 \end{aligned} \tag{46}$$

$$\begin{aligned} & (0.7 \cdot a_1 + a_2) + 0.5 \cdot (0.7 \cdot w_1 + w_2) \geq 78 + 2 \cdot 0.5 \\ & (0.7 \cdot a_1 + a_2) - 0.5 \cdot (0.7 \cdot w_1 + w_2) \geq 78 - 2 \cdot 0.5 \end{aligned} \tag{47}$$

$$\begin{aligned} & (a_2 + 0.5 \cdot w_2) \geq 51 + 3 \cdot 0.5 \\ & (a_2 - 0.5 \cdot w_2) \geq 51 - 3 \cdot 0.5 \end{aligned} \tag{48}$$

The substitute objective function is determined as in the cases of h=0.5. The results are shown in Table 2.

Table 2. Solution for different cases of F.L.P. problem for h=0.5.

h=0,5	Max (a_1+1.5a_2)	Min (w_1+1.5w_2)
a_1	52.00	50.50
w_1	3.00	
a_2	37.00	31.34
w_2	2.00	
z^+	113.50	97.50
z^-	101.50	97.50
a_1+1.5a_2	107.50	97.50
w_1+1.5w_2	6.00	0

4. CONCLUDING REMARKS

Through the presentation of the examples was shown that the proposed results in the use of simple classical formulation without to resort in complicated theories of uncertainty.

Strategic water resources management is searching for optimal solutions for the future base on determinants which are projected with high uncertainty. In the classical formulation of water allocation problems to various users it is customary to use formulations of linear programming in which all the determinants are introduced as crisp numbers.

In this paper an attempt was made to replace the crisp numbers by fuzzy numbers for incorporating the uncertainty of the future conditions. Two fuzzy methodologies were adopted.

The fuzzy methodologies which presented were followed by numerical examples.

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